Q1a

Find the equation of each of these lines in vector form.

(a) The line joining $(3,-2)$ to $(-1,5)$.

A B

[3]

(b) The line passing through $(3,1)$ parallel to $4\mathbf{i}-12\mathbf{j}$.

[2]

$\underline{r} = \overrightarrow{OA} + t\,\overrightarrow{AB}$

ANY VARIABLE (MULTIPLE OF DIRECTION)

DIRECTION VECTOR

POINT ON LINE AS POSITION VECTOR

a)

POINT $\begin{pmatrix}3\\-2\end{pmatrix}$

DIRECTION $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$\begin{pmatrix}-1\\5\end{pmatrix} - \begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}-4\\7\end{pmatrix}$$

$$\boxed{\underline{r} = \begin{pmatrix}3\\-2\end{pmatrix} + t\begin{pmatrix}-4\\7\end{pmatrix}}$$

REMEMBER: INFINITE NUMBER OF CORRECT EQUATIONS

e.g. $\underline{r} = \begin{pmatrix}-1\\5\end{pmatrix} + \lambda\begin{pmatrix}-4\\7\end{pmatrix}$

Q1b

Find the equation of each of these lines in vector form.

(a) The line joining $(3,-2)$ to $(-1,5)$.

[3]

(b) The line passing through $(3,1)$ parallel to $4\mathbf{i}-12\mathbf{j}$.

A $\overrightarrow{AB}$

[2]

$\underline{r} = \overrightarrow{OA} + s\,\overrightarrow{AB}$

ANY VARIABLE (MULTIPLE OF DIRECTION)

DIRECTION VECTOR

POINT ON LINE AS POSITION VECTOR

b)

$\overrightarrow{OA} = \begin{pmatrix}3\\1\end{pmatrix}$

$\overrightarrow{AB} = \begin{pmatrix}4\\-12\end{pmatrix}$

$= \begin{pmatrix}1\\-3\end{pmatrix}$

DIRECTION VECTOR CAN BE SIMPLIFIED

$$\boxed{\underline{r} = \begin{pmatrix}3\\1\end{pmatrix} + s\begin{pmatrix}1\\-3\end{pmatrix}}$$

Q2a

(a) Find the equation of each of these lines in vector form.

(i) Through $(2,-7,9)$ in the same direction as $6\mathbf{i}+3\mathbf{j}-9\mathbf{k}$.

A

(ii) Through $(4,-3,1)$ and $(-8,3,-5)$.

A B

[5]

(b) Show that the point $(10,-6,5)$ does **not** lie on the line found in part (a) (ii).

[2]

a) i) POINT $\overrightarrow{OA}$ $\begin{pmatrix}2\\-7\\9\end{pmatrix}$

DIRECTION $\begin{pmatrix}6\\3\\-9\end{pmatrix} = \begin{pmatrix}2\\1\\-3\end{pmatrix}$

$$\boxed{\underline{r} = \begin{pmatrix}2\\-7\\9\end{pmatrix} + s\begin{pmatrix}2\\1\\-3\end{pmatrix}}$$

ii) POINT $\overrightarrow{OA}$ $\begin{pmatrix}4\\-3\\1\end{pmatrix}$

DIRECTION $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$\begin{pmatrix}-8\\3\\-5\end{pmatrix} - \begin{pmatrix}4\\-3\\1\end{pmatrix} = \begin{pmatrix}-12\\6\\-6\end{pmatrix}$$

SIMPLIFY

$$\boxed{\underline{r} = \begin{pmatrix}4\\-3\\1\end{pmatrix} + t\begin{pmatrix}-2\\1\\-1\end{pmatrix}}$$

Q2b

b) $$\begin{pmatrix}4-2t\\-3+t\\1-t\end{pmatrix} = \begin{pmatrix}10\\-6\\5\end{pmatrix}$$

USING ANY PART SOLVE FOR t

$$1-t=5$$
$$t=-4$$

CHECK IN OTHER PARTS

$4-2(-4)=12 \neq 10$ ✗

$-3+(-4)=-7 \neq -6$ ✗

$$\boxed{(10,-6,5) \text{ DOES NOT LIE ON LINE}}$$

Q3

A and B are the points on the line $\mathbf{r} = \begin{pmatrix}1\\-4\\3\end{pmatrix} + t\begin{pmatrix}2\\a\\-1\end{pmatrix}$ with $t = -1$ and $t = 2$ respectively. Given that the distance from A to B is 9 units, find the possible values of a.

[5]

METHOD 1

SUB IN $t=-1$ $t=2$ FOR $\vec{OA}$ AND $\vec{OB}$

$$\vec{OA} = \begin{pmatrix}1-2\\-4-a\\3+1\end{pmatrix} = \begin{pmatrix}-1\\-4-a\\4\end{pmatrix} \qquad \vec{OB} = \begin{pmatrix}1+4\\-4+2a\\3-2\end{pmatrix} = \begin{pmatrix}5\\-4+2a\\1\end{pmatrix}$$

DISTANCE $|\vec{AB}| = 9$

$$|\vec{AB}| = \sqrt{(5+1)^2 + \left[(\cancel{-4}+2a) - (\cancel{-4}-a)\right]^2 + (1-4)^2}$$

$$= \sqrt{6^2 + (3a)^2 + (-3)^2}$$

$$= \sqrt{45 + (3a)^2} = 9$$

SQUARE BOTH SIDES

$$45 + (3a)^2 = 81$$

$$(3a)^2 = 36$$

$$3a = \pm 6$$

$$\boxed{a = \pm 2}$$

METHOD 2

DIRECTION $\left|\begin{pmatrix}2\\a\\-1\end{pmatrix}\right| = \sqrt{2^2 + a^2 + (-1)^2}$

$$= \sqrt{5 + a^2}$$

DISTANCE $2 - -1 = 3$

$$3\sqrt{5 + a^2} = 9$$

$$5 + a^2 = 3$$

$$5 + a^2 = 9$$

$$a^2 = 4$$

$$\boxed{a = \pm 2}$$

Q4a

a)

$$\vec{BA} = \vec{OA} - \vec{OB}$$

$$\vec{BA} = \begin{pmatrix}2\\-1\\5\end{pmatrix} - \begin{pmatrix}4\\1\\1\end{pmatrix} = \boxed{\begin{pmatrix}-2\\-2\\4\end{pmatrix}}$$

$$\vec{BC} = \vec{OC} - \vec{OB}$$

$$\vec{BC} = \begin{pmatrix}-2\\-5\\9\end{pmatrix} - \begin{pmatrix}4\\1\\1\end{pmatrix} = \boxed{\begin{pmatrix}-6\\-6\\8\end{pmatrix}}$$

Q4b

b) $\cos\theta = \dfrac{a \cdot b}{|a||b|} = \dfrac{\vec{BA} \cdot \vec{BC}}{|\vec{BA}||\vec{BC}|}$

SCALAR PRODUCT

$$\begin{pmatrix} -2 \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ -6 \\ 8 \end{pmatrix} = 12 + 12 + 32 = 56$$

DISTANCE

$$|\vec{BA}| = \sqrt{(-2)^2 + (-2)^2 + 4^2} = \sqrt{24}$$

$$|\vec{BC}| = \sqrt{(-6)^2 + (-6)^2 + 8^2} = \sqrt{136}$$

ANGLE

$$\theta = \cos^{-1}\left(\frac{56}{\sqrt{24}\sqrt{136}}\right)$$

$$= 11.4217\ldots$$

$$\boxed{11.4^\circ \text{ (1dp)}}$$

Q5

The vertices of triangle ABC are the points with coordinates $A(5,-3,0)$, $B(-2,0,-1)$ and $C(1,2,1)$.

(i) Calculate the scalar products $\vec{AB} \cdot \vec{AC}$, $\vec{BA} \cdot \vec{BC}$ and $\vec{CA} \cdot \vec{CB}$.

(ii) What does the result of part (i) tell you about the triangle ABC?

[5]

i) FIND VECTORS E.G $\vec{AB} = \vec{OB} - \vec{OA}$

$$\vec{AB} = \begin{pmatrix} -7 \\ 3 \\ -1 \end{pmatrix} \quad \vec{AC} = \begin{pmatrix} -4 \\ 5 \\ 1 \end{pmatrix}$$

$$\vec{AB} \cdot \vec{AC} = 28 + 15 - 1 = 42$$

$$\boxed{\vec{AB} \cdot \vec{AC} = 42}$$

$\vec{BA} = -\vec{AB}$ (CHANGE SIGN FOR OPPOSITE DIRECTION)

$$\vec{BA} = \begin{pmatrix} 7 \\ -3 \\ 1 \end{pmatrix} \quad \vec{BC} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

$$\vec{BA} \cdot \vec{BC} = 21 - 6 + 2 = 17$$

$$\boxed{\vec{BA} \cdot \vec{BC} = 17}$$

$$\vec{CA} = \begin{pmatrix} 4 \\ -5 \\ -1 \end{pmatrix} \quad \vec{CB} = \begin{pmatrix} -3 \\ -2 \\ -2 \end{pmatrix}$$

$$\vec{CA} \cdot \vec{CB} = -12 + 10 + 2 = 0$$

$$\boxed{\vec{CA} \cdot \vec{CB} = 0}$$

ii) $a \cdot b = 0$ PERPENDICULAR VECTORS

$\cos^{-1}(0) = 90^\circ$

$\boxed{\vec{CA} \cdot \vec{CB} = 0 \text{ SO TRIANGLE ABC MUST BE RIGHTANGLED}}$

Q6

i) $\begin{array}{c} ① \\ ② \\ ③ \end{array} \begin{pmatrix} -7+s \\ -1+s \\ 6-s \end{pmatrix} = \begin{pmatrix} 2+2t \\ 2-t \\ 11+5t \end{pmatrix}$

SOLVE SIMULTANEOUSLY ② + ③

$s = 13 + 4t \qquad 4t = -8 \quad t = -2$

SUB INTO ② FOR S

$-1 + s = 2 + 2 \qquad s = 5$

SUB INTO ①

$-7 + 5 = 2 + 2(-2)$

$-2 = -2$ ✓✓ LINES INTERSECT

SUB IN EITHER S OR t

$\begin{pmatrix} -7+5 \\ -1+5 \\ 6-5 \end{pmatrix}$ OR $\begin{pmatrix} 2+2(-2) \\ 2-(-2) \\ 11+5(-2) \end{pmatrix}$

$\boxed{\text{LINES INTERSECT AT } (-2, 4, 1)}$

ii) $\begin{pmatrix} 3+s \\ -3-2s \\ 1+s \end{pmatrix} = \begin{pmatrix} -1+2t \\ -1+2t \\ 5-3t \end{pmatrix}$ RHS ARE EQUAL ∴ LHS ARE EQUAL

SOLVE LHS

$3+s = -3-2s$

$3s = -6 \qquad s = -2$

SUB IN AND SOLVE FOR t

$3-2 = -1+2t$

$2t = 2 \qquad t = 1$

CHECK IN FINAL EQUATION

$1-2 \neq 5-3$ × ×

LINES DO NOT INTERSECT

CHECK IF DIRECTION VECTORS ARE PARALLEL

$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \neq \mu \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$

$\boxed{\text{DIRECTION VECTORS ARE NOT MULTIPLES THEREFORE LINES ARS SKEW}}$

Q7

[5]

FOR SOME POINT, N, ON LINE

$a \cdot b = 0$ PERPENDICULAR VECTORS

SCALAR $\overrightarrow{NP}$ · DIRECTION $\begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix} = 0$

$\overrightarrow{ON} = \underline{r} = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix} + s\begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix}$ FOR SOME VALUE OF S

$\overrightarrow{NP} = \overrightarrow{OP} - \overrightarrow{ON}$

$$\overrightarrow{NP} = \begin{pmatrix} 6 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} -1+3s \\ 5-6s \\ 2-2s \end{pmatrix} = \begin{pmatrix} 7-3s \\ -5+6s \\ 1+2s \end{pmatrix}$$

SCALAR PRODUCT $= 0$

$$\begin{pmatrix} 7-3s \\ -5+6s \\ 1+2s \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -6 \\ -2 \end{pmatrix} = 0$$

$$3(7-3s) - 6(-5+6s) - 2(1+2s)$$

$$21 - 9s + 30 - 36s - 2 - 4s$$

$$49 - 49s = 0$$

$$s = 1$$

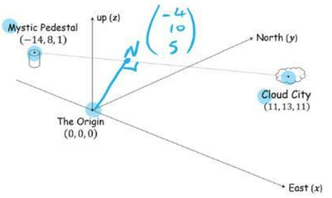
FIND DISTANCE $|\overrightarrow{NP}|$ USING $s=1$

$$|\overrightarrow{NP}| = \left|\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}\right| = \sqrt{4^2+1^2+3^2}$$

$\boxed{\sqrt{26} \text{ UNITS}}$

Q8

In the magical kingdom of Cartesia, all positions are measured relative to the ancient stone of power known as the Origin. This reference system corresponds to the standard x, y, z coordinate system used in mathematics, as shown in the diagram below:



Prince Vector, son of King Prime of Cartesia, needs to fly on his magical unicorn from the top of the Mystic Pedestal all the way to Cloud City, on an urgent rescue mission to save the kingdom from certain doom.

The Mystic Pedestal is 14 miles west and 8 miles north of the Origin, and its top is one mile up from the level of the Origin. Cloud City is 11 miles east and 13 miles north of the Origin, and it is 11 miles up from the level of the Origin.

Since there is not much time, the prince must fly directly from the top of the Mystic Pedestal to Cloud City. Unfortunately, his unicorn's magic levels are low. In order for the unicorn to recharge it must pass within 12 miles of the Origin during the flight, and must do this before reaching the halfway point between the Mystic Pedestal and Cloud City. If the unicorn does not recharge before this point, then it and the prince will crash into the barren wastes, and the kingdom will perish.

After the prince departs, his sister Hypatia (a keen mathematics student and heiress to the throne) remembers that there is a vector method that can be used to determine whether or not her brother's mission will succeed.

Determine whether or not the prince will reach Cloud City successfully, using clear mathematical workings to justify your answer.

[7]

FIND IF SHORTEST (PERPENDICULAR) DISTANCE IS LESS THAN 12 MILES AND LESS THAN HALF WAY

$\underline{r} = \underset{\overrightarrow{OM}}{\text{POINT}} + s\,(\underset{\overrightarrow{MC}}{\text{DIRECTION}})$

POINT $\overrightarrow{OM} = \begin{pmatrix} -14 \\ 8 \\ 1 \end{pmatrix}$

DIRECTION $\overrightarrow{MC} = \overrightarrow{OC} - \overrightarrow{OM}$

$$\begin{pmatrix} 11 \\ 13 \\ 11 \end{pmatrix} - \begin{pmatrix} -14 \\ 8 \\ 1 \end{pmatrix} = \begin{pmatrix} 25 \\ 5 \\ 10 \end{pmatrix}$$

SIMPLIFY

$$\underline{r} = \begin{pmatrix} -14 \\ 8 \\ 1 \end{pmatrix} + s \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$$

N IS A POINT ON LINE PERPENDICULAR TO ORIGIN SO $\overrightarrow{ON} = \underline{r}$ FOR SOME VALUE OF s SUCH THAT $\overrightarrow{ON} \cdot \text{DIRECTION} \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = 0$

PERPENDICULAR $\overrightarrow{ON} \cdot \text{DIRECTION} = 0$

$$\begin{pmatrix} -14 + 5s \\ 8 + s \\ 1 + 2s \end{pmatrix} \cdot \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} = 0$$

$$5(-14 + 5s) + (8 + s) + 2(1 + 2s) = 0$$

$$-70 + 25s + 8 + s + 2 + 4s = 0$$

$$-60 + 30s = 0$$

$$s = \frac{60}{30} = 2$$

USING $s = 2$ FIND $|\overrightarrow{ON}|$

$$|\overrightarrow{ON}| = \left| \begin{pmatrix} -4 \\ 10 \\ 5 \end{pmatrix} \right| = \sqrt{(-4)^2 + 10^2 + 5^2}$$

$$= \sqrt{141} = 11.87$$

$11.87 < 12$ MILES

COMPARE DISTANCES $|\overrightarrow{MN}|$ AND $|\overrightarrow{MC}|$

USING $\overrightarrow{ON} = \begin{pmatrix} -4 \\ 10 \\ 5 \end{pmatrix}$ AND $\overrightarrow{MC} = \begin{pmatrix} 25 \\ 5 \\ 10 \end{pmatrix}$

$$|\overrightarrow{MN}| = \left|\begin{pmatrix} 10 \\ 2 \\ 4 \end{pmatrix}\right| = \sqrt{10^2 + 2^2 + 4^2}$$

$$= \sqrt{120} = 2\sqrt{30}$$

$$|\overrightarrow{MC}| = \left|\begin{pmatrix} 25 \\ 5 \\ 10 \end{pmatrix}\right| = \sqrt{25^2 + 5^2 + 10^2}$$

$$= \sqrt{750} = 5\sqrt{30}$$

$\overrightarrow{MN} = \frac{2}{5}\overrightarrow{MC}$ SO LESS THAN ½ WAY

$\boxed{\text{11.87 < 12 MILES AND } \overrightarrow{MN} = \tfrac{2}{5}\overrightarrow{MC} \text{ PRINCE WILL REACH CLOUD CITY SUCCESSFULLY}}$